On Fuzzy γ-Boundary and Fuzzy γ-Semi Boundary

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Abstract

The aim of this paper is to introduce the concept of fuzzy γ-boundary and fuzzy γ- semi boundary of a fuzzy topological space. Some characterizations are discussed, examples are given and properties are established.

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1.Introduction

The concepts of fuzzy set operations were first introduced by L.A.Zadeh[8] in his paper. After that Chang[3]defined and studied the notion of fuzzy topological space. Pu and Liu[6] defined the notion of fuzzy boundary in fuzzy topological spaces in 1980. Following this, Ahmad and Athar[1] studied their properties. They are also defined the concept of fuzzy semi boundary and discussed their properties. In 2011, Swidi and Oon[5] introduce fuzzy γ-open set and fuzzy γ-closed set and discussed their properties. Usha et al[7] defined the concept of fuzzy γ-semi open sets and fuzzy γ-semi closed sets in fuzzy topological spaces.

Using this, we have introduce fuzzy γ-boundary and fuzzy γ-semi boundary and present several properties of fuzzy γ-boundary and fuzzy γ-semi boundary with proper examples.

2.Preliminaries

For the basic concepts and notations one can refer Chang. The following definitions and lemmas are useful in studying the properties of fuzzy γ-boundary and fuzzy γ-semi boundary.

Definition 2.1: A fuzzy topology is a family τ of fuzzy sets in X which satisfies the following conditions:

(a) $\Phi, X \in \tau$,
(b) If $a, b \in \tau$, then $A \wedge B \in \tau$,
(c) If $A_i \in \tau$ for each $i \in I$, then $\vee_I A_i \in \tau$.
τ is called fuzzy topology for X and the pair (X, τ) is a fuzzy topological space.

Definition 2.2[5]: Let (X, τ) be a fuzzy topological space. A fuzzy subset A of X is called fuzzy γ-open if

$A \leq (\text{int}(\text{cl } A)) \vee \text{cl}(\text{int}(A))$. The complement of a fuzzy γ-open set is called fuzzy γ-closed.

Definition 2.3[5]: Let A be any fuzzy set in the fuzzy topological space (X, τ). Then

$\gamma\text{-cl}(A)=\wedge\{B: B \text{ is fuzzy } \gamma\text{-closed and } B\geq A\}$ and $\gamma\text{-int}(A)=\vee\{B: B \text{ is fuzzy } \gamma\text{-open and } B\leq A\}$.

Lemma 2.4[5]: Let A be any fuzzy set in the fuzzy topological space (X, τ). Then

i. $(\gamma\text{-int}(A))^c = \gamma\text{-cl}(A^c)$ and
ii. $(\gamma\text{-cl}(A))^c = \gamma\text{-int}(A^c)$.

Lemma 2.5[5]: Let A and B be any fuzzy sets in the fuzzy topological space (X, τ). Then the following are true.

(i) γ-cl(A) is fuzzy γ-closed in X,
(ii) γ-int(A) is fuzzy γ-open in X,
(iii) $\gamma\text{-cl}(A) \le \gamma\text{-cl}(B)$ if $A \le B$,
(iv) γ-int(γ-int A) = γ-int A,
(v) γ-cl(γ-cl A) = γ-cl A,
(vi) $\gamma\text{-int}(A \wedge B) = (\gamma\text{-int} A) \wedge (\gamma\text{-int} B)$,
(vii) $\gamma\text{-int}(A \vee B) \ge (\gamma\text{-int} A) \vee (\gamma\text{-int} B)$,
(viii) $\gamma\text{-cl}(A \vee B) = \gamma\text{-cl}(A) \vee \gamma\text{-cl}(B)$ and
(ix) $\gamma\text{-cl}(A \wedge B) \le \gamma\text{-cl}(A) \wedge \gamma\text{-cl}(B)$.

Definition 2.6[7]: Let A be a fuzzy subset of a fuzzy topological space (X,τ). Then A is called fuzzy γ-semi open set of X if there exist a fuzzy γ-open set γ-O such that $\gamma\text{-O} \le A \le cl(\gamma\text{-O})$.

Lemma 2.7[7]: Let (X, τ) be a fuzzy topological space. Then a fuzzy subset A of a fuzzy topological space (X, τ) is fuzzy γ-semi open if and only if $A \le cl(\gamma\text{-int}(A))(A \ge int(\gamma\text{-cl}(A))$.

Definition 2.8[7]: Let (X, τ) be a fuzzy topological space. Then for a fuzzy subset A of X, the fuzzy γ-semi interior of A (briefly γ-sint (A)) is the union of all fuzzy γ-semi open sets of X contained in A. That is, $\gamma\text{-sint}(A) = \vee\{B: B \le A$, B is fuzzy γ-semi open in X}.

Definition 2.9[7]: Let (X, τ) be a fuzzy topological space. Then for a fuzzy subset A of X, the fuzzy γ-semi closure of A (briefly γ-scl (A)) is the intersection of all fuzzy γ-semi closed sets contained in A. That is, $\gamma\text{-scl}(A) = \wedge\{B: B \ge A$, B is fuzzy γ-semi closed}.

Lemma 2.10[7]: Let (X, τ) be a fuzzy topological space. Then for any fuzzy subset A and B of a fuzzy topological space, we have

(i) $\gamma\text{-sint}(A \wedge B) = (\gamma\text{-sint} A) \wedge (\gamma\text{-sint} B)$,

(ii) $\gamma\text{-sint}(A \vee B) \ge (\gamma\text{-sint} A) \vee (\gamma\text{-sint} B)$,

(iii) γ-scl γ-scl(A) = γ-scl(A),

(iv) γ-sint γ-sint(A)= γ-sint(A).

Lemma 2.11[7]: Let (X, τ) be a fuzzy topological space. Then for a fuzzy subset A and B of a fuzzy topological space X, we have

(i) $\gamma\text{-scl}(A \vee B) = \gamma\text{-scl}(A) \vee \gamma\text{-scl}(B)$ and

(ii) $\gamma\text{-scl}(A \wedge B) \le \gamma\text{-scl}(A) \wedge \gamma\text{-scl}(B)$.

Lemma 2.12[7]: Let (X, τ) be a fuzzy topological space. Then for any fuzzy subsets A of X, we have

i. $(\gamma\text{-sint}(A))^c = \gamma\text{-scl}(A^c)$ and
ii. $(\gamma\text{-scl}(A))^c = \gamma\text{-sint}(A^c)$

Lemma 2.13[7]: Let (X, τ) and (Y, σ) be any two fuzzy topological spaces such that X is product related to Y. Then the product $A_1 \times A_2$ of fuzzy γ-open set A_1 of X and a fuzzy γ-open set A_2 of Y is a fuzzy γ-open set of the fuzzy product space X $\times$ Y.

3. Fuzzy γ-Boundary

In this section, we introduce the concept of fuzzy γ-Boundary and their properties are analysed.

Definition 3.1: Let A be a fuzzy set in an fuzzy topological space (X, τ). Then the fuzzy

γ-boundary of A is defined as γ-Bd(A)= γ-cl(A) $\wedge$ γ-cl(A^c).

Obviously γ-Bd(A) is a fuzzy γ-closed set.

Remark 3.2: In fuzzy topology, we have A$\vee$ γ-Bd(A) $\leq$ γ-cl(A), for an arbitrary fuzzy set A in X, the equality need not hold as the following example shows.

Example 3.3: Let X = {a, b} and τ = {0, 1, {$a_{.3}$, $b_{.4}$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {$a_{.7}$, $b_{.6}$}}. Let A = {$a_{.6}$, $b_{.4}$}.

Then γ-cl(A) = {$a_{.7}$, $b_{.4}$} and γ-Bd(A)= {$a_{.5}$, $b_{.4}$}.

It follows that γ-cl(A) $\neq$ {$a_{.6}$, $b_{.4}$}= A$\vee$ γ-Bd(A).

Proposition 3.4: For a fuzzy set A in a fuzzy topological space (X, τ), the following conditions hold.

(1) γ-Bd(A)= γ-Bd(A^C).
(2) If A is fuzzy γ-closed , then γ-Bd(A) $\leq$ A.
(3) If A is fuzzy γ-open , then γ-Bd(A) $\leq$ A^C.
(4) Let A $\leq$ B and B ϵ FγC(X) (resp., B ϵ FγO(X)). Then γ-Bd(A) $\leq$ B (resp.,γ-Bd(A) $\leq$ B^C), where FγC(X) (resp., FγO(X)) denotes the class of fuzzy γ-closed (resp., fuzzy γ-open) sets in X.
(5) (γ-Bd A)^C = γ-int(A) $\vee$ γ-int(A^C).
(6) γ-Bd(A) $\leq$ Bd(A).
(7) γ-cl(γ-Bd(A)) $\leq$ Bd(A).

Proof:

By Definition 3.1, γ-Bd(A) = γ-cl(A) $\wedge$ γ-cl(A^C) and γ-Bd(A^C)= γ-cl(A^C) $\wedge$ γ-cl(A).
Therefore γ-Bd(A) = γ-Bd(A^C). Hence (1).
Let A be fuzzy γ-closed. By Lemma 2.5, γ-cl(A)= A.
γ-Bd(A) $\leq$ γ-cl(A) $\wedge$ γ-cl(A^C) $\leq$ γ-cl(A) = A. Hence (2).
Let A be fuzzy γ-open. By Lemma 2.5, γ-int(A) = A. It follows that
γ-Bd(A) $\leq$ γ-cl(A^C) = [γ-int(A)]^C = A^C. Hence (3).
Let A $\leq$ B. Then by Properties [5]2.7, γ-cl(A) $\leq$ γ-cl(B).
Since B ϵ F γC(X), we have γ-cl(B) = B.
This implies that , γ-Bd(A) = γ-cl(A) $\wedge$ γ-cl(A^C) $\leq$ γ-cl(B) $\wedge$ γ-cl(B^C) $\leq$ γ-cl(B) = B.
That is γ-Bd(A) $\leq$ B.
Let B ϵ FγO(X). Then B^c ϵ FγC(X). Using the above, γ-Bd(A) $\leq$ B^C. Hence (4).
By Definition 3.1, γ-Bd(A)= γ-cl(A) $\wedge$ γ-cl(A^C). Taking complement on both sides, we get
[γ-Bd(A)]^C = [γ-cl(A) $\wedge$ γ-cl(A^C)]^C = [γ-cl(A)]^C $\vee$ [γ-cl(A^C)]^C =γ-int(A^C) $\vee$ γ-int(A).
Hence (5).
Since γ-cl(A) $\leq$ cl(A) and γ-cl(A^C) $\leq$ cl(A^C), we have
γ-Bd(A)= γ-cl(A)$\wedge$ γ-cl(A^C) $\leq$ cl(A) $\wedge$ cl(A^C) = Bd(A). Hence (6).
γ-cl(γ-Bd(A)) = γ-cl(γ-cl(A) $\wedge$ γ-cl(A^C)) $\leq$ γ-cl(γ-cl(A)) $\wedge$ γ-cl(γ-cl(A^C))
= γ-cl(A) $\wedge$ γ-cl(A^C) = γ-Bd(A) $\leq$ Bd(A). Thus γ-cl(γ-Bd(A)) $\leq$ Bd(A). Hence (7).

The converse of (2) and (3) and reverse inequalities of (6) and (7) in the Proposition 3.4 are in general, not true as is shown by the following example .

Example 3.5:(i) Let X = {a, b} and τ = {0, 1, {a.$_2$, b.$_1$},{a.$_8$, b.$_9$}, {a.$_7$, b.$_2$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_8$, b.$_9$},{a.$_2$, b.$_1$}, {a.$_3$, b.$_8$}}. Let A = {a.$_8$, b.$_7$}. Then γ-cl(A) = {a.$_8$, b.$_8$}}and γ-Bd(A)= {a.$_3$, b.$_3$}.

Therefore γ-Bd(A) = {a.$_3$, b.$_3$} $\leq$ A, but A is not fuzzy γ-closed.

(ii) Let X = {a, b} and τ = {0, 1, {a.$_2$, b.$_2$},{a.$_8$, b.$_8$}, {a.$_7$, b.$_7$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_8$, b.$_8$},{a.$_2$, b.$_2$}, {a.$_3$, b.$_3$}}. Let A = {a.$_2$, b.$_3$}.

Then γ-cl(A) = {a.$_3$, b.$_3$}} and γ-Bd(A)= {a.$_3$, b.$_3$}.

Therefore γ-Bd(A)= {a.$_3$, b.$_3$ } $\leq A^C$ = {a.$_8$, b.$_7$}, but A is not fuzzy γ-open.

(iii) Let X = {a, b} and τ = {0, 1, {a.$_3$, b.$_4$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_7$, b.$_6$}}. Let A = {a.$_6$, b.$_4$}. Then γ-cl(A) = {a.$_7$, b.$_4$} and γ-Bd(A) = {a.$_5$, b.$_4$}.

Now Bd(A) = cl(A) $\wedge$ cl(A^C) = {a.$_7$, b.$_6$} $\wedge$ {a.$_7$, b.$_6$} = {a.$_7$, b.$_6$}.Thus γ-Bd(A) $\not\geq$ Bd(A).

Now γ-cl γ-Bd(A)= {a.$_5$, b.$_5$}. This implies that γ-cl γ-Bd(A) $\not\geq$ Bd(A).

Proposition 3.6: Let A be fuzzy set in an fuzzy topological space X. Then

(1) γ-Bd(A) = γ-cl(A) $\wedge$ (γ-int A)^C,

(2) γ-Bd(γ-int(A)) $\leq$ γ-Bd(A),

(3) γ-Bd(γ-cl(A)) $\leq$ γ-Bd(A),

(4) γ-int(A) $\leq$ A $\wedge$ (γ-Bd(A))^C.

Proof:

Since γ-cl(A^C) = (γ-int A)^C, we have γ-Bd(A)= γ-cl(A) $\wedge$ γ-cl(A^C) = γ-cl(A)$\wedge$ (γ-int A)^C. This proves (1).
By Definition 3.1, γ-Bd(γ-int(A)) = γ-cl(γ-int(A)) $\wedge$ γ-cl(γ-int A)^C = γ-cl(γ-int(A)) $\wedge$ γ-cl(γ-cl(A^C))
= γ-cl(γ-sint(A)) $\wedge$ γ-cl(A^C) = γ-cl(γ-int(A)) $\wedge$ (γ-int A)^C $\leq$ γ-cl(A) $\wedge$ (γ-int A)^C
= γ-Bd(A). Hence (2).
γ-Bd(γ-cl(A)) = γ-cl(γ-cl(A))$\wedge$ γ-cl(γ-cl A)^C = γ-cl(γ-cl(A)) $\wedge$ [γ-int(γ-cl A)]^C

$\leq$ γ-cl(A)$\wedge$ (γ-int A)^C = γ-Bd(A). Thus proves (3).

A $\wedge$ (γ-BdA)^C = A $\wedge$ (γ-cl A $\wedge$ γ-cl A^C)^C = A $\wedge$ (γ-int A^C $\vee$ γ-int A)
= (A $\wedge$ γ-int A^C) $\vee$ (A $\wedge$ γ-int A) = (A $\wedge$ γ-int A^C)$\vee$ γ-int(A) $\geq$ γ-int(A). Hence (4).

To show that the inequalities (2), (3) and (4) of Proposition 3.6 are in general irreversible, we have the following example.

Example 3.7: (i) Let X = {a, b} and τ = {0, 1,{a.$_2$, b.$_1$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_8$, b.$_9$}}. Let A = {a.$_3$, b.$_2$}.

Then γ-cl(A)= {a.$_6$, b.$_5$} , γ-int(A)= {a.$_2$, b.$_2$} and calculations give γ-Bd(A) = {a.$_6$, b.$_5$}.

This shows that γ-Bd(A) $\not\leq$ γ-Bd(γ-int(A)) = {a.$_3$, b.$_2$}.

Now we calculate γ-Bd γ-cl(A)= {a.$_3$, b.$_4$}.This shows that γ-Bd(A) $\not\leq$ γ-Bd(γ-cl(A)) and γ-int(A) = {a.$_2$, b.$_2$} $\not\geq$ A $\wedge$ (γ-Bd(A))^C = {a.$_3$, b.$_2$} $\wedge$ {a.$_4$, b.$_5$} = {a.$_3$, b.$_2$}.

Theorem 3.8: Let A and B be a fuzzy sets in an fuzzy topological space (X, τ). Then,

γ-Bd(A $\vee$ B) $\leq$ γ-Bd(A) $\vee$ γ-Bd(B).

Proof:

We use Lemma 2.5 (viii), (ix) to prove this.

γ-Bd(A$\vee$B) = γ-cl(A$\vee$B) $\wedge$ γ-cl(A$\vee$B)^C = γ-cl(A$\vee$B) $\wedge$ γ-cl(A^C $\wedge$ B^C)

$\leq$ (γ-cl(A) $\vee$ γ-cl(B)) $\wedge$ (γ-cl (A ^{C}) $\wedge$ γ-cl (B^C)) $\leq$ (γ-cl(A) $\wedge$ γ-cl (A^C)) $\vee$ (γ-cl(B) $\wedge$ γ-cl(B^C))

= γ-Bd(A)$\vee$ γ-Bd(B). Hence the Proof.

The reverse in equality in Theorem 3.8 is in general not true as shown by the following example.

Example 3.9: Let X = {a, b} and τ = {0, 1, {a.$_3$, b.$_4$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_7$, b.$_6$}}. Let A = {a.$_3$, b.$_6$} and

B = {a.$_4$, b.$_3$}. Then calculations give γ-Bd(A)= {a.$_4$, b.$_5$} and γ-Bd(B)={a.$_6$, b.$_5$}.

Now A $\vee$ B = {a.$_4$, b.$_6$} and γ-Bd(A$\vee$B) = {a.$_5$, b.$_5$}.

This gives that γ-Bd(A) $\vee$ γ-Bd(B) = {a.$_6$, b.$_5$} $\not\leq$ γ-Bd(A $\vee$ B) = {a.$_5$, b.$_5$}.

The following example shows that γ-Bd(A $\wedge$ B) $\not\leq$ γ-Bd(A) $\wedge$ γ-Bd(B) and

γ-Bd(A) $\wedge$ γ-Bd(B) $\not\leq$ γ-Bd(A $\wedge$ B).

Example 3.10: Using the Example 3.9, A = {a.$_3$, b.$_6$} and B = {a.$_4$, b.$_3$}. Then calculations give γ-Bd(A)= {a.$_4$, b.$_5$} and γ-Bd(B) = {a.$_6$, b.$_5$}. Now A $\wedge$ B = {a.$_3$, b.$_3$} and γ-Bd(A $\wedge$ B) = {a.$_5$, b.$_4$}.

This gives that γ-Bd(A) $\wedge$ γ-Bd(B) = {a.$_4$, b.$_5$} $\not\leq$ γ-Bd(A$\wedge$B)= {a.$_5$, b.$_4$} and γ-Bd(A $\wedge$ B) $\not\leq$ γ-Bd(A) $\wedge$ γ-Bd(B).

Theorem 3.11: For any fuzzy sets A and B in an fuzzy topological space (X, τ), one has

γ-Bd(A $\wedge$ B) $\leq$ (γ-Bd(A) $\wedge$ γ-cl(B)) $\vee$ (γ-Bd(B) $\wedge$ γ-cl(A)).

Proof:

We use Lemma 2.5 (viii), (ix) to prove this.

γ-Bd(A $\wedge$ B) = γ-cl(A $\wedge$ B) $\wedge$ γ-cl(A $\wedge$ B)^C = γ-cl(A $\wedge$ B) $\wedge$ γ-cl(A^C $\vee$ B^C)

$\leq$ (γ-cl(A) $\wedge$ γ-cl(B)) $\wedge$ (γ-cl(A^C) $\vee$ γ-cl(B^C))

= (γ-cl(A) $\wedge$ γ-cl(B) $\wedge$ γ-cl(A^C)) $\vee$ (γ-cl(A) $\wedge$$\gamma$-cl(B) $\wedge$ γ-cl(B^C))

= (γ-Bd(A) $\wedge$ γ-cl(B)) $\vee$ (γ-Bd(B) $\wedge$ γ-cl(A)). Hence proved.

Corollary 3.12: For any fuzzy sets A and B in an fuzzy topological space (X, τ), one has

γ-Bd(A $\wedge$ B) $\leq$ γ-Bd(A) $\vee$ γ-Bd(B).

The reverse in equality in Theorem 3.11 is in general not true as shown by the following example.

Example 3.13: Using the Example 3.9, $A = \{a_{.3}, b_{.6}\}$, γ-cl(A)= $\{a_{.4}, b_{.6}\}$, γ-Bd(A)= $\{a_{.4}, b_{.5}\}$ and $B = \{a_{.4}, b_{.3}\}$,

γ-cl(B)= $\{a_{.6}, b_{.5}\}$, γ-Bd(B)= $\{a_{.6}, b_{.5}\}$. Now $A \wedge B = \{a_{.3}, b_{.3}\}$ and γ-Bd($A \wedge B$) = $\{a_{.4}, b_{.4}\}$. This gives that

(γ-Bd(A) $\wedge$ γ-cl(B)) $\vee$ (γ-Bd(B) $\wedge$ γ-cl(A))= $\{a_{.4}, b_{.5}\} \not\leq$ γ-Bd($A \wedge B$)= $\{a_{.4}, b_{.4}\}$.

Proposition 3.14: For any fuzzy sets A in an fuzzy topological space (X, τ), one has

(1) γ-Bd(γ-Bd(A)) $\leq$ γ-Bd(A).
(2) γ-Bd(γ-Bd(γ-Bd(A))) $\leq$ γ-Bd(γ-Bd(A)).

Proof:

We use Lemma 2.5 (v) and Definition 3.1 to prove this.

γ-Bd(γ-Bd(A)) = γ-cl(γ-Bd(A)) $\wedge$ γ-cl(γ-Bd(A))$^C \leq$ γ-cl(γ-Bd(A))
= γ-cl (γ-cl(A) $\wedge$ γ-cl(A^C)) = γ-cl(γ-cl(A)) $\wedge$ γ-cl(γ-cl(A^C))
= γ-cl(A) $\wedge$ γ-cl(A^C) = γ-Bd(A). This proves (1).
γ-Bd(γ-Bd(γ-Bd(A))) = γ-cl(γ-Bd(γ-Bd(A))) $\wedge$ γ-cl(γ-Bd(γ-Bd(A))^C)
= γ-Bd(γ-Bd(A)) $\wedge$ γ-cl(γ-Bd(γ-Bd(A))^C) $\leq$ γ-Bd(γ-Bd(A)). Hence the proof.

The reverse inequality in Theorem 3.14 is in general not true as shown by the following example.

Example 3.15: Let $X = \{a, b\}$ and $\tau = \{0, 1, \{a_{.2}, b_{.2}\}\}$. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is $\tau^c = \{0, 1, \{a_{.8}, b_{.8}\}\}$. Let $A = \{a_{.3}, b_{.4}\}$.

Then calculations give γ-Bd(A)= $\{a_{.7}, b_{.7}\} \not\leq$ γ-Bd(γ-Bd(A)) = $\{a_{.6}, b_{.7}\}$.

Again we calculate γ-Bd(γ-Bd(γ-Bd(A))) =$\{a_{.4}, b_{.4}\}$. This shows that

γ-Bd(γ-Bd(A)) = $\{a_{.6}, b_{.7}\} \not\leq$ γ-Bd(γ-Bd(γ-Bd(A))) =$\{a_{.4}, b_{.4}\}$.

Definition 3.16[4]: If λ is a fuzzy set of (X, τ) and μ is a fuzzy set of (Y,σ) then

$(\lambda \times \mu)(x, y) = \min\{ \lambda(x), \mu(y)\}$, for each $X \times Y$.

Definition 3.17[2]: An fuzzy topological space (X, τ_1) is a product related to an fuzzy topological space (Y, τ_2) if for fuzzy sets A of X and B of Y whenever $C^c \not\geq A$ and $D^c \not\geq B$ implies $C^c \times 1 \vee 1 \times D^c \geq A \times B$, where $C \in \tau_1$ and $D \in \tau_2$, there exist $C_1 \in \tau_1$ and $D_1 \in \tau_2$ such that $C_1{}^c \geq A$ or $D_1{}^c \geq B$ and $C_1{}^c \times 1 \vee 1 \times D_1{}^c = C^c \times 1 \vee 1 \times D^c$.

Lemma 3.18[1]: For fuzzy sets λ, μ, υ and ω in a set S, one has

$(\lambda \wedge \mu) \times (\upsilon \wedge \omega) = (\lambda \times \omega) \wedge (\mu \times \upsilon)$.

Theorem 3.19: Let (X, τ) and (Y,σ) be a fuzzy topological space. If A is a fuzzy subset of a fuzzy topological space (X, τ) and B is a fuzzy subset of a fuzzy topological space (Y,σ). Then

(1) γ-cl A $\times$ γ-cl B $\geq$ γ-cl($A \times B$).
(2) γ-int A $\times$ γ-int B $\leq$ γ-int($A \times B$).

Proof:

By using Definition 3.16, (γ-cl A $\times$ γ-cl B)(x, y) = min{γ-cl A(x), γ-cl B(y)}

$\geq \min\{A(x), B(y)\} = (A \times B)(x, y)$. This shows that γ-cl A $\times$ γ-cl B $\geq$ (x, y).

Thus By Lemma 2.5, γ-cl$(A \times B) \leq \gamma$-cl (γ-cl $A \times \gamma$-cl B) = γ-cl $A \times \gamma$-cl B.

By using Definition 3.16, (γ-int $A \times \gamma$-int B)(x, y)= min{γ-int A(x), γ-int B(y)}

$\leq$ min{A(x), B(y)} = $(A \times B)(x, y)$. This shows that γ-int $A \times \gamma$-int $B \leq (x, y)$.

Thus By Lemma 2.5, γ-int$(A \times B) \geq \gamma$-int (γ-int $A \times \gamma$-int B)=γ-int $A \times \gamma$-int B.

Theorem 3.20: Let (X, τ) and (Y,σ) be fuzzy topological spaces such that X is product related to Y. Then for a fuzzy set A of X and a fuzzy set B of Y γ-cl $(A \times B)$ = γ-cl $A \times \gamma$-cl B.

Proof:

For fuzzy sets A_i's of X and B_j's of Y, we first note that

(i) $\inf\{A_i, B_j\} = \min\{\inf A_i, \inf B_j\}$,
(ii) $\inf\{A_i \times 1\} = (\inf A_i) \times 1$,
(iii) $\inf\{1 \times B_j\} = 1 \times (\inf B_j)$.

In view of above theorem it is sufficient to show that γ-cl$(A \times B) \geq \gamma$-cl $A \times \gamma$-cl B.

Let $A_I \in F\gamma O(X)$ and $B_j \in F\gamma O(Y)$. Then, γ-cl$(A \times B) = \inf\{(A_i \times B_j)^C / (A_i \times B_j)^C \geq A \times B\}$

$= \inf\{A_i^C \times 1 \vee 1 \times B_j^C / A_i^C \times 1 \vee 1 \times B_j^C \geq A \times B\}$

$= \inf\{A_i^C \times 1 \vee 1 \times B_j^C / A_i^C \geq A \text{ or } B_j^C \geq B\}$

$= \min(\inf\{A_i^C \times 1 \vee 1 \times B_j^C / A_i^C \geq A\}, \inf\{A_i^C \times 1 \vee 1 \times B_j^C / B_j^C \geq B\})$.

Since $\inf\{A_i^C \times 1 \vee 1 \times B_j^C / A_i^C \geq A\} \geq \inf\{A_i^C \times 1 / A_i^C \geq A\} = \inf\{A_i^C / A_i^C \geq A\} \times 1$

$=$ γ-cl(A) $\times 1$, $\inf\{A_i^C \times 1 \vee 1 \times B_j^C / A_i^C \geq A\} \geq \inf\{B_j^C \times 1 / B_j^C \geq B\}$

$= 1 \times \inf\{B_j^C / B_j^C \geq B\} = 1 \times$ γ-cl(B).

Thus we have γ-cl$(A \times B) \geq$ min(γ-cl $A \times 1$, $1 \times \gamma$-cl B) = γ-cl $A \times \gamma$-cl B.

Theorem 3.21: Let X_i, i=1, 2, …n be a family of product related fuzzy topological spaces. If each A_i is a fuzzy set in X_i, then

$\gamma\text{-Bd}\prod_{i=1}^{n} A_i = [\gamma\text{-Bd } A_1 \times \gamma\text{-cl } A_2 \times \ldots \times \gamma\text{-cl } A_n] \vee [\gamma\text{-cl } A_1 \times \gamma\text{-Bd } A_2 \times \gamma\text{-cl } A_3 \times \ldots \times$

$\gamma\text{-cl } A_n] \vee \ldots \vee [\gamma\text{-cl } A_1 \times \gamma\text{-cl } A_2 \times \ldots \times \gamma\text{-Bd } A_n]$.

Proof:

It suffices to prove this for n=2, consider γ-Bd$(A_1 \vee A_2)$ = γ-cl$(A_1 \times A_2) \wedge [\gamma\text{-int}(A_1 \times A_2)]^C$

$= (\gamma\text{-cl } A_1 \times \gamma\text{-cl } A_2) \wedge [\gamma\text{-int } A_1 \times \gamma\text{-int } A_2]^C = (\gamma\text{-cl } A_1 \times \gamma\text{-cl } A_2) \wedge [(\gamma\text{-int } A_1 \wedge \gamma\text{-cl } A_1) \times (\gamma\text{-int } A_2 \wedge \gamma\text{-cl } A_2)]^C = (\gamma\text{-cl } A_1 \times \gamma\text{-cl } A_2) \wedge [(\gamma\text{-int } A_1 \times \gamma\text{-cl } A_2) \wedge (\gamma\text{-cl } A_1 \times \gamma\text{-int } A_2]^C$

$= [(\gamma\text{-cl } A_1 \times \gamma\text{-cl } A_2) \wedge (\gamma\text{-int } A_1 \times \gamma\text{-cl } A_2)^C] \vee [((\gamma\text{-cl } A_1 \times \gamma\text{-cl } A_2) \wedge (\gamma\text{-cl } A_1 \times \gamma\text{-int } A_2)^C]$

$= [(\gamma\text{-cl } A_1 \wedge \gamma\text{-int } A_1) \times \gamma\text{-cl } A_2] \vee [\gamma\text{-cl } A_1 \times (\gamma\text{-cl } A_2 \wedge \gamma\text{-int } A_2)]$

$= (\gamma\text{-Bd } A_1 \times \gamma\text{-cl} A_2) \vee (\gamma\text{-cl } A_1 \times \gamma\text{-Bd } A_2)$.

Theorem 3.22: Let f: $(X, \tau) \rightarrow (Y,\sigma)$ be a fuzzy continuous function.

Then γ-Bd(f^{-1}(B)) $\leq$ f^{-1}(γ-Bd(B)), for any fuzzy subset B in Y.

Proof:

Let f be a fuzzy continuous function and B be a fuzzy subset in Y. By using Definition 3.1, we have γ-Bd (f^{-1}(B)) = γ-cl(f^{-1}(B)) $\wedge$ γ-cl(f^{-1}(B))^C= γ-cl(f^{-1}(B))$\wedge$ γ-cl(f^{-1}(B^C)).

Since f is fuzzy continuous and f^{-1}(B) $\leq$ f^{-1}(γ-cl(B)) it follow that γ-cl(f^{-1}(B)) $\leq$ f^{-1}(γ-cl(B)). This together with the above imply that γ-Bd(f^{-1}(B)) $\leq$ f^{-1}(γ-cl(B)) $\wedge$ f^{-1}(γ-cl(B^C))

= f^{-1}(γ-cl(B) $\wedge$ γ-cl(B^C)). That is γ-Bd(f^{-1}(B)) $\leq$ f^{-1}(γ-Bd(B)).

4. Fuzzy γ-Semi Boundary

In this section we define γ-semi boundary and discuss their properties with examples.

Definition 4.1: Let A be a fuzzy set in an fuzzy topological space (X, τ). Then the fuzzy

γ-semi boundary of A is defined as γ-sBd(A)= γ-scl(A)$\wedge$ γ-scl(A^C).

Obviously γ-sBd(A) is a fuzzy γ-semi closed set.

Remark 4.2: In fuzzy topology, we have A$\vee$ γ-sBd(A) $\leq$ γ-scl(A), for an arbitrary fuzzy set A in X, the equality need not hold as the following example shows.

Example 4.3: Let X = {a, b} and τ = {0, 1,{$a_{.8}$, $b_{.8}$}, {$a_{.2}$, $b_{.2}$}, {$a_{.3}$, $b_{.7}$} }. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {$a_{.2}$, $b_{.2}$}, {$a_{.8}$, $b_{.8}$}, {$a_{.7}$, $b_{.3}$}}. Let A = {$a_{.6}$, $b_{.9}$}. Then γ-scl(A) = {$a_{.8}$, $b_{.9}$} and γ-sBd(A)= {$a_{.4}$, $b_{.3}$}. It follows that γ-scl(A) $\neq$ {$a_{.6}$, $b_{.9}$} = A $\vee$ γ-sBd(A).

Proposition 4.4: For a fuzzy set A in a fuzzy topological space (X, τ), the following conditions hold.

(1) γ-sBd(A) = γ-sBd(A^C).
(2) If A is fuzzy γ-semi closed, then γ-sBd(A) $\leq$ A.
(3) If A is fuzzy γ-semi open, then γ-sBd(A) $\leq$ A^C.
(4) Let A $\leq$ B and B ϵ FγSC(X) (resp., B ϵ FγSO(X)). Then γ-sBd(A) $\leq$ B (resp.,γ-sBd(A) $\leq$ B^C), where FγSC(X) (resp., FγSO(X)) denotes the class of fuzzy γ-semi closed (resp., fuzzy γ-semi open) sets in X.
(5) (γ-sBd(A))^C = γ-sint(A) $\vee$ γ-sint(A^C).
(6) γ-sBd(A) $\leq$ Bd(A).
(7) γ-scl(γ-sBd(A)) $\leq$ Bd(A).

Proof:

By Definition 4.1, γ-sBd(A)= γ-scl(A) $\wedge$ γ-scl(A^C) and
γ-sBd(A^C)= γ-scl(A^C) $\wedge$ γ-scl(A). Therefore γ-sBd(A) = γ-sBd(A^C). Hence (1).
Let A be fuzzy γ-semi closed. By Proposition[7] 6.3(ii), γ-scl(A)= A.

γ-sBd(A)= γ-scl(A)$\wedge$ γ-scl(A^C) $\leq$ γ-scl(A) = A. Hence (2).

Let A be fuzzy γ-semi open. By Proposition[7] 5.2(ii), γ-sint(A)= A.
γ-sBd(A) = γ-scl(A) $\wedge$ γ-scl(A^C) $\leq$ γ-scl(A^C) = [γ-sint(A)]^C = A^C. Hence (3).
Let A $\leq$ B. Then by Proposition[7] 6.3(iv), γ-scl(A) $\leq$ γ-scl(B). Since B ϵ F γSC(X), we have γ-scl(B) = B.
This implies that , γ-sBd γ-sBd(A) $\leq$ γ-scl(A) $\leq$ γ-scl(B)=B.
That is γ-sBd(A) $\leq$ B. Let B ϵ FγSO(X). Then B^C ϵ FγSC(X). Then by the above,
γ-sBd(A) $\leq$ B^C. Hence (4).
By Definition 4.1, γ-sBd(A)= γ-scl(A)$\wedge$ γ-scl(A^C). Taking complement on both sides, we get

$[\gamma\text{-sBd}(A)]^C = [\gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C)]^C = [\gamma\text{-scl}(A)]^C \vee [\gamma\text{-scl}(A^C)]^C$
$= \gamma\text{-sint}(A^C) \vee \gamma\text{-sint}(A)$. Hence (5).
Since $\gamma\text{-scl}(A) \leq cl(A)$ and $\gamma\text{-scl}(A^C) \leq cl(A^C)$, then we have
$\gamma\text{-sBd}(A) = \gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C) \leq cl(A) \wedge cl(A^C) = Bd(A)$. Hence (6).
$\gamma\text{-scl}(\gamma\text{-sBd}(A)) = \gamma\text{-scl}(\gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C)) \leq \gamma\text{-scl}(\gamma\text{-scl}(A)) \wedge \gamma\text{-scl}(\gamma\text{-scl}(A^C))$
$= \gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C) = \gamma\text{-sBd}(A) \leq Bd(A)$. Thus $\gamma\text{-scl}(\gamma\text{-sBd}(A)) \leq Bd(A)$.
Hence (7).

The converse of (2) and (3) and reverse inequalities of (6) and (7) in the Proposition 4.4 are in general, not true as is shown by the following example .

Example 4.5: Let $A = \{a_{.6}, b_{.9}\}$ and $B = \{a_{.4}, b_{.1}\}$ in the fuzzy topological space (X, τ) be defined in Example 4.3. Then $\gamma\text{-sBd}(A) = \{a_{.4}, b_{.3}\} \leq A$, but A is not fuzzy γ-semi closed.

$\gamma\text{-sBd}(B) = \{a_{.4}, b_{.3}\} \leq B^C$, but B is not fuzzy γ-semi open.

$Bd(B) = \{a_{.7}, b_{.3}\} \nleq \gamma\text{-sBd}(B) = \{a_{.4}, b_{.3}\}$ and $Bd(B) \nleq \gamma\text{-scl}(\gamma\text{-sBd}(A)) = \{a_{.5}, b_{.3}\}$.

Proposition 4.6: Let A be fuzzy set in an fuzzy topological space (X, τ). Then

(1) $\gamma\text{-sBd}(A) = \gamma\text{-scl}(A) \wedge (\gamma\text{-sint}(A))^C$,

(2) $\gamma\text{-sBd}(\gamma\text{-sint}(A)) \leq \gamma\text{-sBd}(A)$,

(3) $\gamma\text{-sBd}(\gamma\text{-scl}(A)) \leq \gamma\text{-sBd}(A)$,

(4) $\gamma\text{-sint}(A) \leq A \wedge (\gamma\text{-sBd}(A))^C$.

Proof:

Since $\gamma\text{-scl}(A^C) = (\gamma\text{-sint}(A))^C$, we have $\gamma\text{-sBd}(A) = \gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C)$
$= \gamma\text{-scl}(A) \wedge (\gamma\text{-sint}(A))^C$. This proves (1).
$\gamma\text{-sBd}(\gamma\text{-sint}(A)) = \gamma\text{-scl}(\gamma\text{-sint}(A)) \wedge \gamma\text{-scl}(\gamma\text{-sint}(A))^C$
$= \gamma\text{-scl}(\gamma\text{-sint}(A)) \wedge \gamma\text{-scl}(\gamma\text{-scl}(A^C)) = \gamma\text{-scl}(\gamma\text{-sint}(A)) \wedge \gamma\text{-scl}(A^C)$
$= \gamma\text{-scl}(\gamma\text{-sint}(A)) \wedge (\gamma\text{-sint}(A))^C \leq \gamma\text{-scl}(A) \wedge (\gamma\text{-sint}(A))^C = \gamma\text{-sBd}(A)$. Hence (2).
$\gamma\text{-sBd}(\gamma\text{-scl}(A)) = \gamma\text{-scl}(\gamma\text{-scl}(A)) \wedge \gamma\text{-scl}(\gamma\text{-scl}(A))^C$
$= \gamma\text{-scl}(\gamma\text{-scl}(A)) \wedge [\gamma\text{-sint}(\gamma\text{-scl}(A))]^C \leq \gamma\text{-scl}(A) \wedge (\gamma\text{-sint}(A))^C = \gamma\text{-sBd}(A)$.
Thus proves (3).
$A \wedge (\gamma\text{-sBd}(A))^C = A \wedge (\gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C))^C = A \wedge (\gamma\text{-sint}(A^C) \vee \gamma\text{-sint}(A))$
$= (A \wedge \gamma\text{-sint}(A^C)) \vee (A \wedge \gamma\text{-sint}(A)) = (A \wedge \gamma\text{-sint}(A^C)) \vee \gamma\text{-sint}(A) \geq \gamma\text{-sint}(A)$.
Hence (4).

To show that the inequalities (2), (3) and (4) of Proposition 4.6 are in general irreversible, we have the following example.

Example 4.7: Choose $A = \{a_{.4}, b_{.1}\}$ in the fuzzy topological space X defined in Example 4.3. Then calculations give $\gamma\text{-sint}(A) = \{a_{.1}, b_{.1}\}$ and

$\gamma\text{-sBd}(A) = \{a_{.4}, b_{.3}\} \nleq \gamma\text{-sBd}(\gamma\text{-sint}(A)) = \{a_{.2}, b_{.1}\}$.

The following example shows that $\gamma\text{-sBd}(A) \nleq \gamma\text{-sBd}(\gamma\text{-scl}(A))$.

Example 4.8: Let $X = \{a, b\}$ and $\tau = \{0, 1, \{a_{.2}, b_{.1}\}, \{a_{.8}, b_{.8}\}\}$. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is $\tau^c = \{0, 1, \{a_{.8}, b_{.9}\}, \{a_{.2}, b_{.2}\}\}$. Let $A = \{a_{.6}, b_{.3}\}$, then calculations give $\gamma\text{-scl}(A) = \{a_{.6}, b_{.5}\}$ and

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γ-sBd(A) = {a.$_6$, b.$_5$} $\not\leq$ γ-sBd(γ-scl(A)) = {a.$_4$, b.$_5$}.

The following example shows that A $\wedge$ (γ-sBd(A))^C $\not\leq$ γ-sint(A).

Example 4.9: Let X = {a, b} and τ = {0, 1,{a.$_8$, b.$_8$}, {a.$_2$, b.$_2$}, {a.$_3$, b.$_7$} }. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_2$, b.$_2$}, {a.$_8$, b.$_8$}, {a.$_7$, b.$_3$}}. Let A = {a.$_4$, b.$_1$}. Then γ-sint(A) = {a.$_2$, b$_0$} and γ-sBd(A)= {a.$_4$, b.$_3$}. It follows that A $\wedge$ [γ-sBd(A)]^C = {a.$_4$, b.$_1$} $\not\leq$ γ-sint(A).

Theorem 4.10: Let A and B be a fuzzy sets in an fuzzy topological space (X, τ). Then,

γ-sBd(A $\vee$ B) $\leq$ γ-sBd(A) $\vee$ γ-sBd(B).

Proof:

We use Lemma 2.11 to prove this.

γ-sBd(A $\vee$ B) = γ-scl(A $\vee$ B) $\wedge$ γ-scl(A $\vee$ B)^C = γ-scl(A $\vee$ B) $\wedge$ γ-scl(A^C $\wedge$ B^C)

$\leq$ (γ-scl(A) $\vee$ γ-scl(B)) $\wedge$ (γ-scl(A^C) $\wedge$ γ-scl (B^C))

$\leq$ (γ-scl(A) $\wedge$ γ-scl(A^C)) $\vee$ (γ-scl(B) $\wedge$ γ-scl(B^C)) = γ-sBd(A)$\vee$ γ-sBd(B).

Hence the Proof.

The reverse in equality in Theorem 4.10 is in general not true as shown by the following example.

Example 4.11: Let X = {a, b} and τ = {0, 1, {a.$_8$, b.$_8$}, {a.$_2$, b.$_1$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_2$, b.$_2$}, {a.$_8$, b.$_9$}}. Let A = {a.$_4$, b.$_1$} and B = {a.$_3$, b.$_4$}. Then calculations give γ-sBd(A) = {a.$_4$, b.$_3$} and

γ-sBd(B) = {a.$_3$, b.$_6$}. Now A $\vee$ B = {a.$_4$, b.$_4$} and γ-sBd(A $\vee$ B) = {a.$_4$, b.$_5$}. This gives that

γ-sBd(A) $\vee$ γ-sBd(B) = {a.$_4$, b.$_6$} $\not\leq$ γ-sBd(A $\vee$ B) = {a.$_4$, b.$_5$}.

The following example shows that γ-sBd(A $\wedge$ B) $\not\leq$ γ-sBd(A) $\wedge$ γ-sBd(B) and

γ-sBd(A) $\wedge$ γ-sBd(B) $\not\leq$ γ-sBd(A $\wedge$ B).

Example 4.12: Let X = {a, b} and τ = {0, 1, {a.$_8$, b.$_6$}, {a.$_2$, b.$_3$}}. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is τ^c = {0, 1, {a.$_2$, b.$_4$}, {a.$_8$, b.$_7$}}. Let A = {a.$_3$, b.$_4$} and B = {a.$_6$, b.$_2$}. Then calculations give γ-sBd(A)= {a.$_3$, b.$_5$} and

γ-sBd(B) ={a.$_5$, b.$_4$}. Now A $\wedge$ B = {a.$_3$, b.$_2$} and γ-sBd(A $\wedge$ B)= {a.$_4$, b.$_3$}. This gives that

γ-sBd(A) $\wedge$ γ-sBd(B) = {a.$_3$, b.$_4$} $\not\leq$ γ-sBd(A $\wedge$ B) = {a.$_4$, b.$_3$} and

γ-sBd(A $\wedge$ B) $\not\leq$ γ-sBd(A) $\wedge$ γ-sBd(B).

Theorem 4.13: For any fuzzy sets A and B in an fuzzy topological space (X, τ), one has

γ-sBd(A $\wedge$ B) $\leq$ (γ-sBd(A) $\wedge$ γ-scl(B)) $\vee$ (γ-sBd(B) $\wedge$ γ-scl(A)).

Proof:

We use Lemma 2.11 to prove this.

γ-sBd(A $\wedge$ B) = γ-scl(A $\wedge$ B) $\wedge$ γ-scl(A $\wedge$ B)^C = γ-scl(A $\wedge$ B) $\wedge$ γ-scl(A^C $\vee$ B^C)

$\leq (\gamma\text{-scl}(A) \wedge \gamma\text{-scl}(B)) \wedge (\gamma\text{-scl}(A^C) \vee \gamma\text{-scl}(B^C))$

$= (\gamma\text{-scl}(A) \wedge \gamma\text{-scl}(B) \wedge \gamma\text{-scl}(A^C)) \vee (\gamma\text{-scl}(A) \wedge \gamma\text{-scl}(B) \wedge \gamma\text{-scl}(B^C))$

$= (\gamma\text{-sBd}(A) \wedge \gamma\text{-scl}(B)) \vee (\gamma\text{-sBd}(B) \wedge \gamma\text{-scl}(A))$. Hence proved.

Corollary 4.14: For any fuzzy sets A and B in an fuzzy topological space X, one has

$\gamma\text{-sBd}(A \wedge B) \leq \gamma\text{-sBd}(A) \vee \gamma\text{-sBd}(B)$.

The reverse in equality in Theorem 4.13 is in general not true as shown by the following example.

Example 4.15: Let X = {a, b} and $\tau = \{0, 1, \{a_{.3}, b_{.2}\}, \{a_{.6}, b_{.8}\}\}$. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is $\tau^c = \{0, 1, \{a_{.7}, b_{.8}\}, \{a_{.4}, b_{.2}\}\}$. Let $A = \{a_{.4}, b_{.6}\}$ and $B = \{a_{.5}, b_{.3}\}$. Then calculations give $\gamma\text{-sBd}(A) = \{a_{.5}, b_{.4}\}$,

$\gamma\text{-scl}(A) = \{a_{.5}, b_{.6}\}$, $\gamma\text{-scl}(B) = \{a_{.5}, b_{.4}\}$ and $\gamma\text{-sBd}(B) = \{a_{.5}, b_{.4}\}$.

Now $A \wedge B = \{a_{.4}, b_{.3}\}$ and $\gamma\text{-sBd}(A \wedge B) = \{a_{.4}, b_{.4}\}$. This gives that

$(\gamma\text{-sBd}(A) \wedge \gamma\text{-scl}(B)) \vee (\gamma\text{-sBd}(B) \wedge \gamma\text{-scl}(A)) = \{a_{.5}, b_{.4}\} \nleq \gamma\text{-sBd}(A \wedge B) = \{a_{.4}, b_{.4}\}$.

Proposition 4.16: For any fuzzy sets A in an fuzzy topological space (X, τ), one has

(1) $\gamma\text{-sBd}(\gamma\text{-sBd}(A)) \leq \gamma\text{-sBd}(A)$.
(2) $\gamma\text{-sBd}(\gamma\text{-sBd}(\gamma\text{-sBd}(A))) \leq \gamma\text{-sBd}(\gamma\text{-sBd}(A))$.

Proof:

$\gamma\text{-sBd}(\gamma\text{-sBd}(A)) = \gamma\text{-scl}(\gamma\text{-sBd}(A)) \wedge \gamma\text{-scl}(\gamma\text{-sBd}(A))^C$
$\leq \gamma\text{-scl}(\gamma\text{-sBd}(A)) = \gamma\text{-scl}(\gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C)) = \gamma\text{-scl}(\gamma\text{-scl}(A)) \wedge \gamma\text{-scl}(\gamma\text{-scl}(A^C))$
$= \gamma\text{-scl}(A) \wedge \gamma\text{-scl}(A^C) = \gamma\text{-sBd}(A)$. This proves (1).
$\gamma\text{-sBd}(\gamma\text{-sBd}(\gamma\text{-sBd}(A))) = \gamma\text{-scl}(\gamma\text{-sBd}(\gamma\text{-sBd}(A))) \wedge \gamma\text{-scl}(\gamma\text{-sBd}(\gamma\text{-sBd}(A))^C)$
$= \gamma\text{-sBd}(\gamma\text{-sBd}(A)) \wedge \gamma\text{-scl}(\gamma\text{-sBd}(\gamma\text{-sBd}(A))^C) \leq \gamma\text{-sBd}(\gamma\text{-sBd}(A))$. Hence the proof.

The reverse inequality in Theorem 4.16 is in general not true as shown by the following example.

Example 4.17: Let X = {a, b} and $\tau = \{0, 1, \{a_{.3}, b_{.2}\}, \{a_{.6}, b_{.8}\}\}$. Then (X, τ) is a fuzzy topological space. The family of all fuzzy closed sets of τ is $\tau^c = \{0, 1, \{a_{.7}, b_{.8}\}, \{a_{.4}, b_{.2}\}\}$. Let $A = \{a_{.4}, b_{.6}\}$.Then calculations give

$\gamma\text{-sBd}(A) = \{a_{.5}, b_{.7}\} \nleq \gamma\text{-sBd}(\gamma\text{-sBd}(A)) = \{a_{.5}, b_{.6}\}$.

It follows that $\gamma\text{-sBd}(\gamma\text{-sBd}(A)) = \{a_{.5}, b_{.6}\} \nleq \gamma\text{-sBd}(\gamma\text{-sBd}(\gamma\text{-sBd}(A))) = \{a_{.5}, b_{.5}\}$.

Theorem 4.18: Let (X, τ) be a fuzzy topological space. If A is a fuzzy subset of a fuzzy topological space X and B is a fuzzy subset of a fuzzy topological space Y. Then

(1) $\gamma\text{-scl } A \times \gamma\text{-scl } B \geq \gamma\text{-scl}(A \times B)$

(2) $\gamma\text{-sint } A \times \gamma\text{-sint } B \leq \gamma\text{-sint } (A \times B)$.

Proof:

By using Definition 3.16, $(\gamma\text{-scl } A \times \gamma\text{-scl } B)(x, y) = \min\{\gamma\text{-scl } A(x), \gamma\text{-scl } B(y)\}$

$\geq \min\{A(x), B(y)\} = (A \times B)(x, y)$. This shows that $\gamma\text{-scl } A \times \gamma\text{-scl } B \geq (x, y)$.

Thus By Lemma 2.10, γ-scl(A × B) ≤ γ-scl (γ-scl A × γ-scl B) = γ-scl A × γ-scl B.

By using Definition 3.16, (γ-sint A × γ-sint B)(x, y) = min{γ-sint A(x), γ-sint B(y)}

≤ min{A(x), B(y)} = (A × B)(x, y). This shows that γ-sint A × γ-sint B ≤ (x, y).

Thus By Lemma 2.10, γ-sint(A × B) ≥ γ-sint (γ-sint A × γ-sint B) = γ-sint A × γ-sint B.

Theorem 4.19: Let X and Y be fuzzy topological spaces such that x is product related to Y. Then for a fuzzy set A of X and a fuzzy set B of Y γ-scl (A × B) = γ-scl A × γ-scl B.

Proof:

For fuzzy sets A_i's of X and B_j's of Y, we first note that

(i) $\inf\{ A_i , B_j\}= \min\{\inf A_i , \inf B_j\}$,
(ii) $\inf\{ A_i \times 1\}= (\inf A_i)\times 1$,
(iii) $\inf\{1\times B_j \}= 1\times (\inf B_j)$.

In view of above theorem it is sufficient to show that γ-scl(A × B) ≥ γ-scl A × γ-scl B.

Let $A_i \in F\gamma SO(X)$ and $B_j \in F\gamma SO(Y)$. Then, $\gamma\text{-scl}(A \times B) = \inf\{(A_I \times B_j)^C /(A_i \times B_j)^C$

$\geq A \times B \}= \inf\{A_i^C \times 1 \vee 1 \times B_j^C / A_i^C \times 1 \vee 1 \times B_j^C \geq A \times B \}$

$= \inf\{ A_i^C \times 1 \vee 1 \times B_j^C /A_i^C \geq A \text{ or } B_j^C \geq B \}$

$= \min (\inf \{ A_i^C \times 1 \vee 1 \times B_j^C /A_i^C \geq A \}, \inf \{A_i^C \times 1 \vee 1 \times B_j^C /B_j^C \geq B \})$.

Since $\inf\{A_i^C \times 1 \vee 1 \times B_j^C /A_i^C \geq A \} \geq \inf\{A_i^C \times 1 /A_i^C \geq A \}$

$= \inf\{ A_i^C /A_i^C \geq A \} \times 1 = \gamma\text{-cl}(A) \times 1, \inf \{A_i^C \times 1 \vee 1 \times B_j^C / A_i^C \geq A \}$

$\geq \inf \{B_j^C \times 1 /B_j^C \geq B \} = 1 \times \inf\{ B_j^C / B_j^C \geq B \}= 1 \times \gamma\text{-cl}(B)$.

Thus we have γ-scl(A × B) ≥ min(γ-scl A × 1, 1 × γ-scl B) = γ-scl A × γ-scl B.

Theorem 4.20: Let X_i, i=1, 2, …n be a family of product related fuzzy topological spaces. If each A_i is a fuzzy set in X_i, then

$\gamma\text{-sBd}\prod_{i=1}^{n} A_i= [\gamma\text{-sBd}(A_1) \times \gamma\text{-scl}(A_2) \times \ldots \times \gamma\text{-scl}(A_n)] \vee [\gamma\text{-scl}(A_1) \times \gamma\text{-sBd}(A_2) \times \gamma\text{-scl}(A_3) \times \ldots \times \gamma\text{-scl}(A_n)] \vee \ldots \vee [\gamma\text{-scl}(A_1) \times \gamma\text{-scl}(A_2) \times \ldots \times \gamma\text{-sBd}(A_n)]$.

Proof:

It suffices to prove this for n=2, consider $\gamma\text{-sBd}(A_1 \vee A_2) = \gamma\text{-scl}(A_1 \times A_2) \wedge [\gamma\text{-sint}(A_1 \times A_2)]^C = (\gamma\text{-scl}(A_1) \times \gamma\text{-scl} (A_2)) \wedge [\gamma\text{-sint}(A_1) \times \gamma\text{-sint}(A_2)]^C$

$= (\gamma\text{-scl}(A_1) \times \gamma\text{-scl} (A_2)) \wedge [(\gamma\text{-sint}(A_1) \wedge \gamma\text{-scl}(A_1)) \times (\gamma\text{-sint}(A_2) \wedge \gamma\text{-scl}(A_2))]^C$

$= (\gamma\text{-scl}(A_1) \times \gamma\text{-scl} (A_2)) \wedge [(\gamma\text{-sint}(A_1) \times \gamma\text{-scl}(A_2)) \wedge (\gamma\text{-scl}(A_1) \times \gamma\text{-sint}(A_2)]^C$

$= [(\gamma\text{-scl}(A_1) \times \gamma\text{-scl} (A_2)) \wedge (\gamma\text{-sint}(A_1) \times \gamma\text{-scl}(A_2))^C] \vee [((\gamma\text{-scl}(A_1) \times \gamma\text{-scl} (A_2)) \wedge(\gamma\text{-scl}(A_1) \times \gamma\text{-sint}(A_2))^C] = [(\gamma\text{-scl}(A_1) \wedge \gamma\text{-sint}(A_1)) \times \gamma\text{-scl}(A_2)] \vee [\gamma\text{-scl}(A_1) \times (\gamma\text{-scl}(A_2) \wedge \gamma\text{-sint}(A_2))]$

$= (\gamma\text{-sBd}(A_1) \times \gamma\text{-scl}(A_2)) \vee (\gamma\text{-scl}(A_1) \times \gamma\text{-sBd}(A_2))$.

Theorem 4.21: Let f: (X, τ) $\rightarrow$ (Y,σ) be a fuzzy continuous function.

Then γ-sBd(f^{-1}(B)) $\leq$ f^{-1}(γ-sBd(B)), for any fuzzy subset B in Y.

Proof:

Let f be a fuzzy continuous function and B be a fuzzy subset in Y. By using Definition 4.1, we have γ-sBd (f^{-1}(B)) = γ-scl(f^{-1}(B)) $\wedge$ γ-scl(f^{-1}(B))^C= γ-scl(f^{-1}(B))$\wedge$ γ-scl(f^{-1}(B^C)).

Since f is fuzzy continuous and f^{-1}(B) $\leq$ f^{-1}(γ-scl(B)) it follow that γ-scl(f^{-1}(B)) $\leq$ f^{-1}(γ-scl(B)). This together with the above imply that γ-sBd(f^{-1}(B)) $\leq$ f^{-1}(γ-scl(B)) $\wedge$ f^{-1}(γ-scl(B^C)) = f^{-1}(γ-scl(B) $\wedge$ γ-scl(B^C)).

That is γ-sBd(f^{-1}(B)) $\leq$ f^{-1}(γ-sBd(B)).

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